

Heavy flavor diffusion in weakly coupled $\mathcal{N} = 4$ Super Yang-Mills theory

Paul M. Chesler and Aleksi Vuorinen

*Department of Physics, University of Washington,
Seattle, WA 98195-1560, U.S.A.*

E-mail: pchesler@u.washington.edu, vuorinen@phys.washington.edu

ABSTRACT: We use perturbation theory to compute the diffusion coefficient of a heavy quark or scalar moving in $\mathcal{N} = 4$ $SU(N_c)$ Super Yang-Mills plasma to leading order in the coupling and the ratio $T/M \ll 1$. The result is compared both to recent strong coupling calculations in the same theory and to the corresponding weak coupling result in QCD. Finally, we present a compact and simple formulation of the Lagrangian of our theory, $\mathcal{N} = 4$ SYM coupled to a massive fundamental $\mathcal{N} = 2$ hypermultiplet, which is well-suited for weak coupling expansions.

KEYWORDS: Extended Supersymmetry, Thermal Field Theory.

JHEP11(2006)037

Contents

1. Introduction	1
2. The setup	3
2.1 $\mathcal{N} = 4$ SYM with massive quarks	3
2.2 Diffusion of a heavy non-relativistic particle	4
3. Calculations and results	5
4. Discussion	9
5. Conclusions and future directions	12
A. The Lagrangian	13
B. Matrix elements and integrals	16
B.1 Matrix elements	16
B.2 Integrals	16

1. Introduction

An important and interesting challenge facing theorists investigating heavy ion physics is to predict the rate of energy loss of a heavy quark moving through a quark gluon plasma (QGP), which is a quantity of direct experimental relevance [1–3]. For weak coupling and ultrarelativistic quarks, $\gamma v \gg 1/g \gg 1$, the dominant mechanism for this is gluon bremsstrahlung, while for the experimentally equally important region $\gamma v \lesssim 1$, the energy loss occurs through elastic collisions with light plasma constituents. Both of these cases have been extensively studied [4–8], but all existing calculations share the fundamental shortcoming that they assume the plasma to be weakly coupled, which need not be the case in the energy range of interest for instance for RHIC physics.

While one has grown to rely upon lattice QCD as the source of direct information on the strong coupling regime of various static observables, it is at present still a relatively inefficient tool in the description of real-time phenomena (for recent advances, see e.g. ref. [9]). A lot of attention has therefore been turned towards addressing the question of the energy loss rate of a quark moving in a strongly coupled non-Abelian plasma in an entirely different framework, the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with gauge group $SU(N_c)$. There, one has the unique opportunity of being able to access analytically the strong coupling limit of the quantum field theory (in its large N_c limit) via the famous AdS/CFT conjecture that relates the theory to dual type IIB supergravity

on an $AdS_5 \times S^5$ background [10]. The energy loss calculation then reduces to studying classical string dynamics in the AdS_5 background, which has yielded useful analytical results for the heavy (and light) quark energy loss parameters in the strong coupling limit (see refs. [11–17] and references therein).

While the development of non-equilibrium lattice field theory methods as well as the search for dual string theories of more QCD-like theories continue, it is worthwhile to first ask the more modest question of what kind of qualitative, or even quantitative, insight the QCD community can draw from the existing SYM calculations. An obvious way of addressing this is to perform similar weak coupling calculations in the SYM theory that have been carried out in the QCD context and compare the results on one hand to the the weak coupling limit of QCD and on the other hand to the strong coupling limit of the $\mathcal{N} = 4$ SYM. This can be expected to yield valuable information on the similarities and differences of the two theories and to furthermore indicate, to which extent one can simply extrapolate the existing weak coupling results in QCD to strong coupling. In the present paper, we aim to do exactly this by investigating the simplest observable related to the energy loss of a non-relativistic heavy quark in the SYM theory, its diffusion coefficient, in the weak coupling regime. The calculation is to a large extent parallel to the corresponding QCD computation of Moore and Teaney [8] and generalizes many of its results to the SYM case.

Our paper is organized as follows. In section 2, we briefly introduce $\mathcal{N} = 4$ SYM and write down its Lagrangian in a form useful for weak coupling expansions, after which we discuss when and how one may use semi-classical kinetic theory techniques to obtain the diffusion coefficient of a heavy particle in a quantum field theory. In section 3, we then go through the necessary calculations, after which we display our main result for the heavy quark diffusion coefficient, D_Q , that is to be further analyzed and discussed in section 4. In section 5, we finally draw conclusions and outline some future work to be carried out through weak coupling kinetic theory calculations in the SYM theory. The appendices contain further computational details, such as a relatively detailed derivation of our Lagrangian as well as a discussion of how the necessary scattering amplitudes and integrals were evaluated.

Throughout the paper, we will for convenience use notation adopted from refs. [8, 18]. This implies working with the Minkowski space metric $-+++$ and denoting four-vectors by capital letters P, Q , three-vectors by bold ones \mathbf{p}, \mathbf{q} and the absolute values of the latter by p, q . The Dirac gamma matrices are defined so that γ_0 is anti-hermitian while the γ_i are hermitian, and consistently with this we have $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$ and $\beta \equiv i\gamma_0$. The gauge is fixed by choosing to work in the Coulomb gauge.

For the most part, we will work with four-component Majorana spinors that can be written in the special form

$$\psi = \begin{pmatrix} e\zeta^* \\ \zeta \end{pmatrix}, \tag{1.1}$$

where $e \equiv i\sigma_2$ and ζ denotes a two-component Weyl fermion. For the Majorana spinors,

$$\bar{\psi} \equiv \psi^\dagger \beta = \psi^T \epsilon \gamma_5, \tag{1.2}$$

with ϵ being the 4×4 matrix

$$\epsilon = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}.$$

2. The setup

2.1 $\mathcal{N} = 4$ SYM with massive quarks

The theory we consider is $\mathcal{N} = 4$ Super Yang-Mills coupled to a $\mathcal{N} = 2$ heavy fundamental hypermultiplet. The field content of $\mathcal{N} = 4$ SYM consists of a gauge field A_μ , four Majorana fermions ψ_i and three complex scalars ϕ_p , while the additional $\mathcal{N} = 2$ multiplet is composed of two heavy scalars Φ_n and a Dirac fermion ω . All $\mathcal{N} = 4$ fields transform under the adjoint representation of the gauge group $SU(N_c)$ and are therefore traceless hermitian $N_c \times N_c$ matrices, while the $\mathcal{N} = 2$ sector consists of N_c component vectors in color space transforming under the fundamental representation.

Following ref. [19], we define $\phi_p = 1/\sqrt{2}(X_p + iY_p)$, with X_p and Y_p hermitian, which allows us to write our Lagrangian in the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \quad (2.1)$$

with

$$\begin{aligned} \mathcal{L}_0 = & -\text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_i \not{D} \psi_i + (DX_p)^2 + (DY_p)^2 \right\} \\ & - \Phi_n^\dagger (-D^2 + M^2) \Phi_n - \bar{\omega} (\not{D} + M) \omega, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \mathcal{L}_1/g = & \text{tr} \left\{ -i\bar{\psi}_i \alpha_{ij}^p [X_p, \psi_j] + \bar{\psi}_i \gamma_5 \beta_{ij}^p [Y_p, \psi_j] \right\} - \bar{\omega} (Y_1 - i\gamma_5 X_1) \omega \\ & + 2\sqrt{2} \text{Im} \left(-\bar{\omega} P_+ \psi_1 \Phi_1 - \Phi_2^\dagger \bar{\psi}_1 P_+ \omega + \Phi_1^\dagger \bar{\psi}_2 P_+ \omega - \bar{\omega} P_+ \psi_2 \Phi_2 \right) \\ & - 2M \Phi_n^\dagger Y_1 \Phi_n, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \mathcal{L}_2/g^2 = & -\frac{1}{2} \text{tr} (i[\chi_A, \chi_B])^2 + (-1)^n \Phi_n^\dagger \left([\phi_2, \phi_2^\dagger] + [\phi_3, \phi_3^\dagger] \right) \Phi_n \\ & - 4\text{Re} \left(\Phi_1^\dagger [\phi_2, \phi_3] \Phi_2 \right) - \frac{1}{2} |(-1)^n \Phi_n^\dagger t_a \Phi_n|^2 - 2 |\Phi_2^\dagger t_a \Phi_1|^2 \\ & - \Phi_n^\dagger \{ \phi_1, \phi_1^\dagger \} \Phi_n. \end{aligned} \quad (2.4)$$

Here, D denotes covariant derivatives in the appropriate representations of $SU(N_c)$, $\chi \equiv (X_1, Y_1, X_2, Y_2, X_3, Y_3)$ and a sum over repeated indices is implied. The matrices α^p and β^p are given by

$$\alpha^1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad (2.5a)$$

$$\beta^1 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad \beta^2 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \quad (2.5b)$$

and they satisfy the algebra

$$\begin{aligned} \{\alpha^p, \alpha^q\} &= \{\beta^p, \beta^q\} = -2\delta^{pq}, \\ [\alpha^p, \beta^q] &= 0. \end{aligned} \tag{2.6}$$

For more details on the derivation of this Lagrangian, see appendix A.

From the form of the Lagrangian it is clear that neither the heavy fermion nor the heavy scalar number is independently conserved, as eq. (2.3) fails to be invariant under the separate global U(1) transformations

$$\begin{aligned} \Phi_i &\rightarrow U_\Phi \Phi_i, \\ \omega &\rightarrow U_\omega \omega, \end{aligned} \tag{2.7}$$

where U_Φ and U_ω are independent phases. This implies that the diffusion coefficients for heavy quarks and heavy scalars are in general not independently well-defined. However, if $U_\Phi = U_\omega$, the transformation of eq. (2.7) does leave the Lagrangian invariant, which means that this combined transformation gives rise to a conserved heavy flavor current that includes both fermions and scalars. It is the diffusion of this heavy flavor density that we are interested in.

2.2 Diffusion of a heavy non-relativistic particle

Following the approach of ref. [8], let us consider the kinematics of a heavy particle immersed in weakly coupled $\mathcal{N} = 4$ SYM plasma at temperature T . We assume that the particle is in thermal equilibrium with the plasma and that its mass $M \gg T$, in which case the typical energy of all types of excitations is $E \sim T$ and the typical momentum of the heavy particle is $p \sim \sqrt{MT}$. At weak coupling, the dominant scattering processes for the heavy particles are $2 \leftrightarrow 2$ elastic collisions with light plasma constituents, in which the typical momentum exchanged is $q \sim T$ and the typical change in the heavy particle velocity is $\delta v \sim T/M \ll 1$. It thus takes many collisions for the velocity to change significantly, and consequently the collisions may be treated as uncorrelated events, in which the heavy particles receive random kicks from the medium. In addition, the mean free path of the heavy particle $\lambda_{\text{MFP}} \sim \left(\frac{M}{T}\right)^2 \frac{1}{g^4 T \log \frac{1}{g}}$ is parametrically large in comparison with its de Broglie wavelength, thus allowing one to use semiclassical methods in the treatment of its dynamics.

Let us then look at the trajectory of a heavy particle that starts from the origin at $t = 0$ and denote its position at time t by $\mathbf{x}(t)$. Under the assumption that collisions with the medium force the heavy particle to undergo a random walk, the diffusion coefficient D is defined by

$$\langle x^2(t) \rangle = 6Dt. \tag{2.8}$$

Denoting the random force exerted on the particle by the medium by $\xi(t)$, the above assumptions imply that the dynamics of the particle follow from the Langevin equation (see e.g. ref. [20])

$$\frac{dp_i}{dt} = \xi_i(t) - \mu p_i, \tag{2.9}$$

where μ is the momentum drag coefficient. As collisions with the light particles are uncorrelated, we furthermore have

$$\langle \xi_i(t) \xi_j(t') \rangle = \kappa \delta_{ij} \delta(t - t'), \quad (2.10)$$

where 3κ is the mean squared momentum transfer per unit time. Using the equilibrium relation $\langle p^2 \rangle = 3MT$ as well as the solution to the above differential equation, it is then easy to show that [8]

$$\mu = \frac{\kappa}{2MT}, \quad \langle x^2(t) \rangle = \frac{6Tt}{M\mu}, \quad (2.11)$$

from which it follows that the heavy particle diffusion coefficient is given by

$$D = \frac{2T^2}{\kappa}. \quad (2.12)$$

The above result relating the diffusion coefficient to κ proves highly useful for our purposes, as in the semiclassical regime where kinetic theory is valid we may immediately write down an expression for the latter in terms of the scattering amplitudes of the quantum theory. Denoting the heavy particles by H , the light particles by ℓ and the Bose and Fermi distribution functions by $n_b(k)$ and $n_f(k)$, respectively, the mean squared momentum transfer per unit time is given by [8, 21]

$$3\kappa = \frac{1}{16(2\pi)^5 M^2} \int \frac{d^3\mathbf{k} d^3\mathbf{k}' d^3\mathbf{p}'}{k_0 k'_0} \delta^3(\mathbf{p} - \mathbf{p}' + \mathbf{k}' - \mathbf{k}) \delta(k - k') \\ \times \sum_{H\ell, H'\ell'} \left\{ |\mathcal{M}_{H\ell \rightarrow H'\ell'}|^2 n_\ell(k) (1 \pm n_{\ell'}(k')) \right\}. \quad (2.13)$$

Here, $|\mathcal{M}_{H\ell \rightarrow H'\ell'}|^2$ stands for the scattering amplitudes squared — summed over all internal degrees of freedom of the light particles and the final state heavy particle and averaged over those of the initial heavy particle (including the flavors) — for the process¹ $H\ell \rightarrow H'\ell'$. The plus sign is taken when $\ell' = b$ represents a final state boson and the minus sign when $\ell' = f$ represents a final state fermion.

3. Calculations and results

In the non-relativistic limit where $M \gg T$, the number of interaction terms in the Lagrangian relevant for the scattering of massive particles can be greatly reduced, as there exists a hierarchy in the M dependence of the various scattering amplitudes. First of all, the amplitude for any tree level process that contains an intermediate heavy particle will be suppressed by an inverse power of M relative to those with an intermediate light particle. Second, in the non-relativistic limit each external heavy fermion² will introduce a factor of

¹The field theory also allows for heavy particles to scatter off of other heavy particles, but their contribution to the integral of κ is suppressed by an exponential of M/T .

²We use the convention $\bar{u}_s(\mathbf{p}) u_{s'}(\mathbf{p}) = 2M \delta_{ss'}$ in the normalization of the heavy spinors.

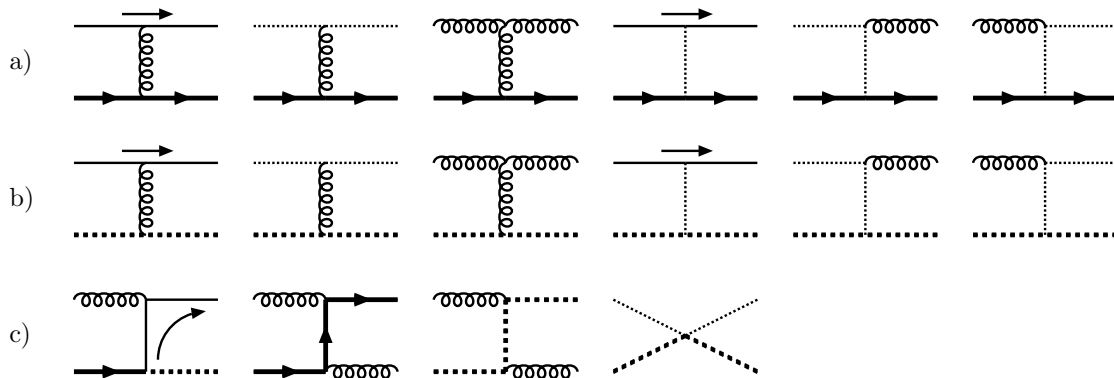


Figure 1: a) The lowest order tree-level processes contributing to the heavy quark diffusion coefficient, the amplitudes of which come with the maximal power of M . b) The corresponding diagrams relevant for the heavy scalar diffusion coefficient. c) Examples of processes contributing to the heavy flavor diffusion coefficient but suppressed by powers of T/M . The solid and dotted bold lines correspond to the heavy quarks and scalars, respectively, the solid and dotted light lines to the corresponding massless fields, and the wavy lines to gluons. The arrows drawn adjacent to the light quark lines indicate the (arbitrary) direction of the flow of the Majorana fermion number (see e.g. ref. [22]).

\sqrt{M} to the amplitudes, and each heavy scalar/gluon vertex (with momentum $P \sim (M, \mathbf{0})$ flowing through it) as well as each heavy scalar/light scalar vertex will introduce an explicit factor of M . Therefore, at leading order in g the diagrams proportional to the highest power of M are those, in which a heavy fermion or scalar scatters elastically off of a light plasma constituent via the exchange of a light intermediate boson. These processes are depicted in figure 1.a-b, while some examples of processes whose amplitudes are suppressed by positive powers of T/M are shown in figure 1.c. The latter include inelastic processes, diagrams with heavy intermediate lines and graphs containing a four-scalar vertex that is independent of M .

Finally, we note that in the non-relativistic limit we may neglect the coupling of X_p to heavy fermions, which follows from the fact that $\bar{\omega}\gamma_5\omega$ is parity odd. In the non-relativistic limit, the scattering of fermions via scalar exchange is independent of spin and thus conserves the parity of the fermions, implying that the heavy fermion scattering amplitudes containing an exchanged X_p must be suppressed by inverse powers of M . In summary, to obtain the desired heavy flavor diffusion coefficient to leading order in T/M and g , we may neglect \mathcal{L}_2 entirely and replace \mathcal{L}_1 by the effective interaction Lagrangian

$$\mathcal{L}_1^{\text{eff}}/g = \text{tr} (\bar{\psi}_i \gamma_5 \beta_{ij}^1 [Y_1, \psi_j]) - \bar{\omega} Y_1 \omega - 2M \Phi_n^\dagger Y_1 \Phi_n. \quad (3.1)$$

After neglecting the species changing Yukawa terms from \mathcal{L}_1 , the Lagrangian becomes invariant under the separate $U(1)$ transformations of eq. (2.7). It then follows that to leading order in M , the heavy fermion and scalar currents are independently conserved, and therefore the corresponding fermion and scalar diffusion coefficients D_Q and D_S can

be independently defined. For these currents, the mean squared momentum transfer per unit time is given by

$$3\kappa_{\text{H}} = \frac{1}{16(2\pi)^5 M^2} \int \frac{d^3\mathbf{k} d^3\mathbf{k}' d^3\mathbf{p}'}{k_0 k'_0} \delta^3(\mathbf{p} - \mathbf{p}' + \mathbf{k}' - \mathbf{k}) \delta(k - k') \times \sum_{\ell, \ell'} \left\{ |\mathcal{M}_{H\ell \rightarrow H\ell'}|^2 n_{\ell}(k) (1 \pm n_{\ell'}(k')) \right\}, \quad (3.2)$$

where H stands for either Q or S, and the corresponding diffusion coefficients read

$$D_{\text{H}} = \frac{2T^2}{\kappa_{\text{H}}}. \quad (3.3)$$

Even though D_{Q} and D_{S} are independently well defined, a quick inspection of the forms of \mathcal{L}_0 and $\mathcal{L}_1^{\text{eff}}$ reveals that their values are in fact equal. To see this, note that if the momentum exchanged in a collision is $\mathbf{q} = \mathbf{p}' - \mathbf{p} \sim T$, where \mathbf{p} and \mathbf{p}' are the momenta of the incoming and outgoing heavy quarks, respectively, then the spinors $u_s(\mathbf{p})$ corresponding to the external legs satisfy (up to $\mathcal{O}(T/M)$ corrections)

$$\begin{aligned} \bar{u}_s(\mathbf{p}) u_{s'}(\mathbf{p}') &\approx 2M \delta_{ss'}, \\ \bar{u}_s(\mathbf{p}) \gamma_{\mu} u_{s'}(\mathbf{p}') &\approx -2iM \delta_{ss'} \delta_{\mu 0}, \end{aligned} \quad (3.4)$$

implying that the contributions of the heavy fermions to scattering amplitudes are spin independent. To obtain the tree level scattering amplitudes that we are interested in (and that are not sensitive to particle statistics), we may therefore replace the heavy quark by a complex scalar field Σ , with the factors of $\pm 2M$ added explicitly to the corresponding vertices. The couplings of Σ to the light fields are then identical to those of Φ_n , and therefore the corresponding scattering amplitudes agree as well. In what follows, we will exploit this symmetry and only consider the heavy fermion diffusion coefficient.

Before we can proceed to the actual computation of D_{Q} , we must still deal with the fact that the expression for κ_{Q} given in eq. (3.2) is infrared sensitive, which can be seen by noting that if one were to use bare propagators for the exchanged bosons in the scattering amplitudes squared, the resulting integrals in eq. (3.2) would diverge in the $q \rightarrow 0$ limit. This can be attributed to the long range potentials associated with the exchange of the massless bosons which, however, are modified by the interactions with the plasma that cut the divergences off at the scale gT . Taking the interactions into account, the IR problem is naturally dealt with by including self energy corrections to the corresponding propagators. As kinematics furthermore require that the energy exchanged in a collision be suppressed relative to the spatial momentum by a factor of $\sqrt{T/M}$, we note that the appropriate self energy corrections are those due to static thermal screening. We can therefore simply add static screening masses to the propagators in question, and taking into account the fact that only the temporal gluon propagator enters the calculations, obtain as the required resummed propagators

$$D_{ab}^{00}(\mathbf{p}) = -\frac{1}{p^2 + m_{\text{D}}^2} \delta_{ab}, \quad (3.5)$$

$$G_{ab}^p(\mathbf{p}) = \frac{1}{p^2 + m_{\text{S}}^2} \delta_{ab} \quad (3.6)$$

for A_0 and ϕ_p , respectively.

The squared scattering amplitudes for the processes shown in figure 1.a are computed in appendix B. Denoting heavy quarks by Q , light fermions, scalars and gluons by f , s and g , respectively, and initial and final light particle three-momenta by \mathbf{k} and \mathbf{k}' , the results read

$$|\mathcal{M}_{Qf \rightarrow Qf}|^2 = 32g^4 d_A M^2 k^2 (1 + \cos \theta) \frac{1}{(q^2 + m_D^2)^2} + 32g^4 d_A M^2 k^2 (1 - \cos \theta) \frac{1}{(q^2 + m_S^2)^2}, \quad (3.7a)$$

$$|\mathcal{M}_{Qs \rightarrow Qs}|^2 = 48g^4 d_A M^2 k^2 \frac{1}{(q^2 + m_D^2)^2}, \quad (3.7b)$$

$$|\mathcal{M}_{Qg \rightarrow Qg}|^2 = 8g^4 d_A M^2 k^2 (1 + \cos^2 \theta) \frac{1}{(q^2 + m_D^2)^2}, \quad (3.7c)$$

$$|\mathcal{M}_{Qs \rightarrow Qg}|^2 = 8g^4 d_A M^2 k^2 \sin^2 \theta \frac{1}{(q^2 + m_S^2)^2}, \quad (3.7d)$$

$$|\mathcal{M}_{Qg \rightarrow Qs}|^2 = 8g^4 d_A M^2 k^2 \sin^2 \theta \frac{1}{(q^2 + m_S^2)^2}, \quad (3.7e)$$

where $d_A \equiv N_c^2 - 1$ and θ is the angle between \mathbf{k} and \mathbf{k}' . These expressions have been summed over all internal degrees of freedom of the light particles as well as over those of the final state heavy quark, and averaged over those of the initial state heavy quark.

Appendix B shows how to evaluate the integrals appearing in eq. (3.2). At weak coupling, where one may use the leading order results (see e.g. ref. [19])

$$m_D^2 = 2g^2 N_c T^2, \quad (3.8)$$

$$m_S^2 = g^2 N_c T^2, \quad (3.9)$$

consistency in the weak coupling expansion requires that the terms in the integrals proportional to positive powers of m_D/T or m_S/T be neglected, which allows us to carry out the integrations analytically. Using the relation of eq. (3.3) between κ_Q and D_Q , we then obtain as our main result

$$D_Q = \frac{12\pi}{d_A g^4 T} \left\{ \log \frac{2T}{m_D} + \frac{13}{12} - \gamma_E + \frac{1}{3} \log 2 + \frac{\zeta'(2)}{\zeta(2)} \right\}^{-1}, \quad (3.10)$$

to which the leading corrections come in at relative order $O(g)$. This result is independent of the scalar screening mass, which follows from the fact that as $\cos \theta = 1 - \frac{q^2}{2k^2}$, every term in eqs. (3.7a)–(3.7e) containing m_S is infrared safe and therefore does not diverge in the limit $m_S \rightarrow 0$. Also, one should take note of the fact that due to the equality of the heavy quark and heavy scalar diffusion coefficients, the more general heavy flavor diffusion coefficient D , given by the average of the two, coincides with eq. (3.10) as well. To better facilitate a comparison with the strong coupling limit of the theory — in which only the heavy flavor diffusion coefficient is *a priori* well-defined — we will in the following sections refer only to the latter quantity also in the weak coupling context.

4. Discussion

Having obtained an expression for the heavy quark (flavor) diffusion coefficient in weakly coupled $\mathcal{N} = 4$ SYM theory, it is interesting to analyze it and to compare it on one hand to the strong coupling result of Herzog *et al.* and others [11, 12] and on the other hand to the corresponding weak coupling calculation in QCD by Moore and Teaney [8]. An immediate observation from both the expanded result of eq. (3.10) and the integral of eq. (3.2) is that when written in terms of the 't Hooft coupling $\lambda = g^2 N_c$, the only explicit dependence on N_c in the results comes from an overall factor of $(1 - 1/N_c^2)^{-1}$. Keeping in mind that this multiplicative factor can be reintroduced to the results at any later time, we shall in what follows, unless explicitly stated otherwise, consider the large N_c limit and set $(1 - 1/N_c^2)^{-1} = 1$.

In order to inspect the domain of validity of the small m_D/T and m_S/T expansions, we plot in figure 2 our result for $1/(DT)$ obtained both with and without the expansion, with the curve for the latter originating from a numerical evaluation of the integral in eq. (3.2), in which the weak coupling expressions for the screening masses given in eqs. (3.8) and (3.9) are used. We observe that the two curves begin to differ significantly at $\lambda \sim 3/4$, and that the expanded result for D in fact starts to diverge when $\lambda \gtrsim 2$. This unphysical behavior signals the breakdown of the leading order weak coupling expansion for D and consequently implies that higher order corrections must be taken into account to gain even qualitative information about the intermediate coupling regime. Not performing the small screening mass expansion amounts to doing a partial resummation of our results, where some higher order contributions are included, but others, such as those coming from additional processes like bremsstrahlung or from corrections to the scattering amplitudes or screening masses, are neglected. While it is not *a priori* obvious that this is sufficient to gain quantitative information about the behavior of D at larger coupling, it is clear from figure 2 that including these corrections improves the qualitative behavior of the diffusion coefficient, as the unphysical divergence of D at $\lambda \approx 3$ is removed. In what follows, we will therefore not use the small screening mass expansion, but instead evaluate the integrals of eq. (3.2) numerically.

In figure 3.a, we investigate the behavior of our result at stronger coupling by plotting $1/(DT)$ as a function of λ , with the screening masses still given by their leading order weak coupling expressions. As λ increases, an increasingly important source of ambiguity in this plot comes from neglecting the NLO corrections to the screening masses, which are expected to be sizable already at $\lambda \sim 1$.³ Therefore, one must exercise caution in interpreting these results and should preferably only use them when $\lambda \ll 10$. In figure 3.b, we have circumvented this problem by plotting $D\lambda^2$ as a function of m_D/T , with m_S^2/m_D^2 fixed, but have now replaced it with an ambiguity related to the choice of these ratios at any $\lambda \gtrsim 1$. From this figure it is, however, evident that the diffusion coefficient is relatively

³At these couplings, the leading order weak coupling results for the screening masses yield values $\gtrsim T$. For consistency, one should therefore use resummed propagators in their evaluation, which would result in important (but at present unknown) $\mathcal{O}(g^3)$ correction terms. For a discussion of this topic in the case of QCD, see ref. [23].

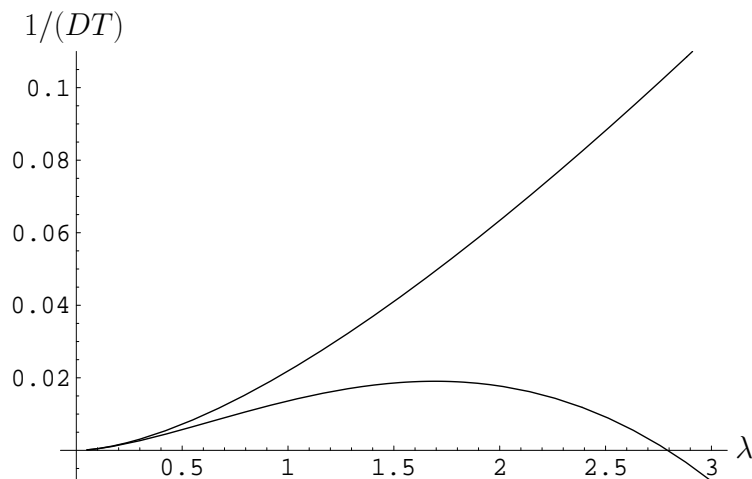


Figure 2: Plots of our two weak coupling results for $1/(DT)$: the expanded version taken from eq. (3.10) (lower curve) and the one obtained via a numerical evaluation of the integral in eq. (3.2) (upper curve). The large N_c limit has been taken here, and the weak coupling expressions for the screening masses have been used.

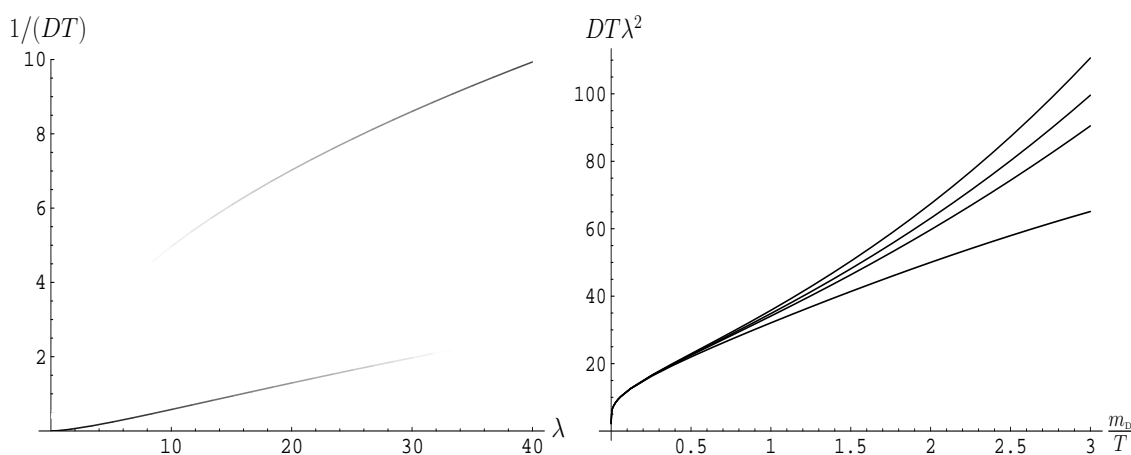


Figure 3: Left: A plot of the weak (lower curve) and strong (upper curve) coupling results for $1/(DT)$ in the large N_c limit. The weak coupling curve has been obtained via numerical integration of eq. (3.2), with the screening masses given by their leading order perturbative expressions, while the strong coupling curve is taken from eq. (4.1). Right: The value of the (integrated) weak coupling result for $DT\lambda^2$ as a function of m_D/T for $m_S^2/m_D^2 = 0, 1/4, 1/2$ and 1 (from bottom to top) in the large N_c limit.

insensitive to deviations of the ratio m_D^2/m_S^2 from the leading order weak coupling result of $\frac{1}{2}$.

For reasons of comparison, we have included already in figure 3.a a plot of the strong

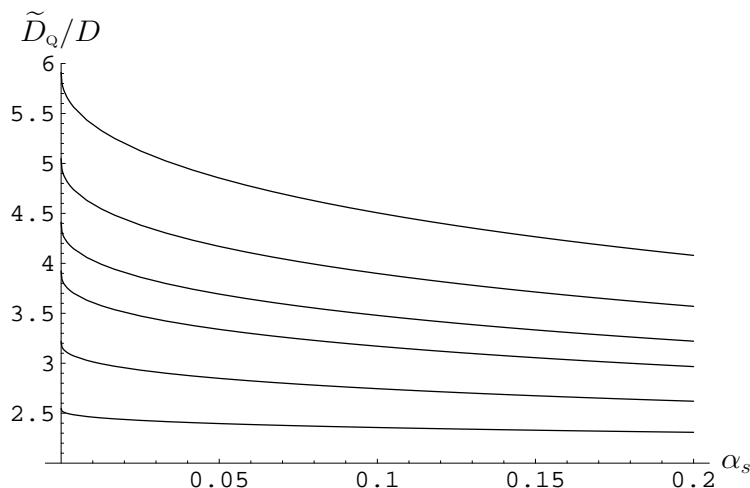


Figure 4: Equal coupling plots of \tilde{D}_Q/D for $N_c = 3$ and $N_f = 0, 1, 2, 3, 4$ and 8 (from top to bottom) as functions of α_s .

coupling result for $1/(DT)$ as obtained from refs. [11, 12], according to which

$$D = \frac{2}{\pi\sqrt{\lambda}T} \tag{4.1}$$

at large values of λ .⁴ As can be seen from figure 3.a, it is easy to find a smooth, monotonic interpolating function that has the correct limiting behavior at small and large λ , but at intermediate values of the coupling there is a wide region where neither result offers an accurate quantitative estimate for D . At $\lambda \sim 20$, the weak coupling extrapolation of D is seen to be roughly six times larger than the corresponding strong coupling prediction, which is not surprising as we in any case are far beyond the region of validity of the weak coupling result here. However, we note that upon comparing the forms of the weak and strong coupling curves at intermediate couplings, it appears likely that the strong coupling expression yields an underestimate for the diffusion coefficient in this region.

To put our discussion on a more quantitative footing, let us recall that using their weak coupling result, Moore and Teaney estimated the heavy quark diffusion coefficient for $N_f = 3$ QCD to be $\tilde{D}_Q \approx 1/T$ at $\alpha_s = 0.5$, where we have adopted the convention of denoting QCD quantities with tildes. In order to convert our result for the $\mathcal{N} = 4$ diffusion coefficient at $\alpha = 0.5$ into at least a rough estimate for this quantity, we refer to the behavior of the ratio \tilde{D}_Q/D at weak (and equal) coupling, as shown in figure 4. There, D is obtained by numerically integrating eq. (B.3), with the screening masses given by their weak coupling expressions at $N_c = 3$, while \tilde{D}_Q is evaluated from the equivalent QCD expressions [8]. We see from this plot that at least at weak coupling, the ratio is a slowly

⁴The region of validity of this result is currently somewhat unclear, but the authors of ref. [11] point out that at the experimentally interesting couplings of $\lambda \sim 20$, it is already expected to obtain sizable corrections. Therefore, we urge the reader to use this expression only for $\lambda \gg 20$.

decreasing function of α_s , and assuming this trend to carry on to stronger couplings, we estimate an upper bound $\tilde{D}_Q/D \lesssim 3$ at $\alpha_s = \alpha = 0.5$. Keeping in mind that the strong coupling result of eq. (4.1) is likely to be an overestimate at these couplings (corresponding to $\lambda \approx 19$), we on the other hand obtain $D|_{\alpha=0.5} \gtrsim 1/(7T)$, which translates into the rough estimate $\tilde{D}_Q \sim 3/(7T) \sim 1/(2T)$ at $\alpha_s = 0.5$.

Finally, from the point of view of the above comparisons between $\mathcal{N} = 4$ SYM theory and QCD, it is also of some interest to investigate the ratio of the diffusion coefficients of the two theories at very weak coupling to get some insight into the order of magnitude of this quantity. Setting the couplings of the two theories again equal, we have from ref. [8]

$$\begin{aligned} \tilde{D}_Q = \frac{72\pi}{d_A g_s^4 T} & \left\{ \log \frac{2T}{\tilde{m}_D} + \frac{1}{2} - \gamma_E + \frac{\zeta'(2)}{\zeta(2)} \right. \\ & \left. + \frac{N_f}{2N_c} \left(\log \frac{4T}{\tilde{m}_D} + \frac{1}{2} - \gamma_E + \frac{\zeta'(2)}{\zeta(2)} \right) \right\}^{-1}, \end{aligned} \quad (4.2)$$

from which we obtain at asymptotically weak coupling (where the logs in eqs. (3.10) and (4.2) dominate over the constant terms)

$$\frac{\tilde{D}_Q}{D} \rightarrow \frac{6}{1 + \frac{N_f}{2N_c}}. \quad (4.3)$$

We observe that for all reasonable values of N_f , the QCD diffusion coefficient is considerably larger than that of $\mathcal{N} = 4$ SYM, which is mostly a reflection of the fact that there are more light degrees of freedom in weakly coupled $\mathcal{N} = 4$ SYM theory for the heavy quark to scatter off of. For example, in the $N_f = 0$ case of pure Yang-Mills theory (in which all light particles are in the adjoint representation), one has 16 light bosonic degrees of freedom, while $\mathcal{N} = 4$ SYM contains 64 bosonic and 64 fermionic degrees of freedom. A straightforward analysis shows that each bosonic degree of freedom contributes equally to the leading log in κ_Q , while each fermionic degree of freedom contributes half as much, so that at asymptotically weak coupling the diffusion coefficient of pure Yang-Mills theory should be $\frac{64+32}{16} = 6$ times bigger than that of $\mathcal{N} = 4$ SYM, just as we observed above. Similar conclusions have been drawn also in ref. [24], where the authors compared the weak coupling results for the shear viscosity in $\mathcal{N} = 4$ SYM and QCD.

5. Conclusions and future directions

In the paper at hand, we have investigated the diffusion of a heavy, non-relativistic thermal particle — either a quark or a scalar belonging to a fundamental $\mathcal{N} = 2$ hypermultiplet — immersed in $\mathcal{N} = 4$ Super Yang-Mills plasma. We have derived a result for the heavy flavor diffusion coefficient that is valid to leading order in g and T/M , and compared it to the corresponding strong coupling results of refs. [11, 12] as well as to the weak coupling calculations of ref. [8] in QCD. Our findings show that a naive extrapolation of the weak coupling result to intermediate couplings yields a relatively large disagreement with the

strong coupling predictions, while in the weak coupling limit the heavy flavor diffusion coefficient in the SYM theory is considerably smaller than the corresponding QCD quantity. Based on our analysis, we have estimated the heavy quark diffusion coefficient in QCD to be roughly $\tilde{D}_Q \sim 1/(2T)$ at $\alpha_s = 0.5$.

As is evident from the small number of weak coupling results available in $\mathcal{N} = 4$ SYM theory, especially in comparison with the strong coupling limit or with perturbative QCD, there is a lot of further work to be done that can provide the QCD community useful insights from the abundance of existing AdS/CFT calculations. The obvious next goal related to the present work — and one that that should be straightforward to achieve — is to generalize the results of this paper to the case of the diffusion of a relativistic quark with $\gamma v \gtrsim 1$. This is work in progress.

Acknowledgments

We are grateful to Chris Herzog, Andreas Karch and Larry Yaffe for suggesting the topic and for their valuable comments and advice, and to Andy O’Bannon, Guy Moore, Maurizio Piai, Dam Son and Matt Strassler for useful discussions. This work was supported by the U.S. Department of Energy under Grant No. DE-FG02-96ER40956.

A. The Lagrangian

In this first appendix, we aim to present a somewhat detailed derivation of the Lagrangian of our theory, $\mathcal{N} = 4$ Super Yang-Mills with a massive $\mathcal{N} = 2$ hypermultiplet, following to a large extent the treatment of ref. [18]. The field content of the $\mathcal{N} = 4$ theory consists of one gauge multiplet with components (A_μ, λ, D) and three chiral multiplets χ , χ' and χ'' with components $(\phi, \psi, \mathcal{F})$, $(\phi', \psi', \mathcal{F}')$ and $(\phi'', \psi'', \mathcal{F}'')$, respectively, while the $\mathcal{N} = 2$ sector contains two fundamental massive chiral multiplets Q' and Q'' with components (Φ', Ψ', F') and (Φ'', Ψ'', F'') . Here, A_μ is an $SU(N_c)$ gauge field, λ , ψ , ψ' , ψ'' , Ψ' and Ψ'' are Majorana fermions, ϕ , ϕ' , ϕ'' , Φ' , and Φ'' are complex scalars, and D , \mathcal{F} , \mathcal{F}' , \mathcal{F}'' , F' and F'' so-called auxiliary fields. All fields in the $\mathcal{N} = 4$ sector transform in the adjoint representation of the gauge group and are hence traceless, hermitian $N_c \times N_c$ matrices, while the $\mathcal{N} = 2$ fields are fundamental under $SU(N_c)$ and can therefore be viewed as N_c -component vectors in color space.

If we fix the heavy particle masses to M , the superpotential of the theory reads

$$f(\chi, \chi', \chi'', Q', Q'') = -i\sqrt{2} Q''^T \chi Q' + 2i\sqrt{2} \text{tr}(\chi[\chi', \chi'']) + M Q'^T Q''. \quad (\text{A.1})$$

Integrating out the auxiliary fields and going through some straightforward algebra, we obtain the Lagrangian [18]⁵

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \quad (\text{A.3})$$

⁵In doing this, we have identified and corrected several misprints in the original reference. These are: an extra second term on the third-to last row of eq. (27.4.1), a missing ϵ matrix between λ and ψ on the second row of eq. (27.9.3), reversed indices m and n in the second term on the tenth row of eq. (27.9.33) and several misprints in the μ -dependent terms of eq. (27.9.33). The last two rows of the latter equation

where

$$\begin{aligned}
 \mathcal{L}_0 = & -\text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + 2D_\mu \phi^\dagger D^\mu \phi + \bar{\psi} \not{D} \psi + \bar{\lambda} \not{D} \lambda + 2D_\mu \phi'^\dagger D^\mu \phi' + 2D_\mu \phi''^\dagger D^\mu \phi'' \right. \\
 & + \bar{\psi}' \not{D} \psi' + \bar{\psi}'' \not{D} \psi'' \left. \right\} - D_\mu \Phi'^\dagger D^\mu \Phi' - M^2 \Phi'^\dagger \Phi' - D_\mu \Phi''^\dagger D^\mu \Phi'' \\
 & - M^2 \Phi''^\dagger \Phi'' - \frac{1}{2} \bar{\Psi}' \not{D} \Psi' - \frac{1}{2} \bar{\Psi}'' \not{D} \Psi'' - 2M \text{Re} \bar{\Psi}' P_+ \Psi'',
 \end{aligned} \tag{A.4}$$

$$\begin{aligned}
 \mathcal{L}_1/g = & \text{Im} \left\{ -4\sqrt{2} \text{tr} \bar{\lambda} P_+ [\phi^\dagger, \psi] + 4\sqrt{2} \text{tr} \bar{\psi}' P_+ [\phi, \psi''] + 4\sqrt{2} \text{tr} \bar{\psi}'' P_+ [\phi', \psi] \right. \\
 & - 4\sqrt{2} \text{tr} \bar{\psi}'' P_+ [\phi'', \psi] + 4\sqrt{2} \text{tr} \bar{\psi}' P_+ [\phi'^\dagger, \lambda] + 4\sqrt{2} \text{tr} \bar{\psi}'' P_+ [\phi''^\dagger, \lambda] \\
 & - 2\sqrt{2} \bar{\Psi}'' P_+ \psi \Phi' - 2\sqrt{2} \Phi''^T \bar{\psi} P_+ \Psi' + 2\sqrt{2} \Phi'^\dagger \bar{\lambda} P_+ \Psi' - 2\sqrt{2} \bar{\Psi}'' P_+ \lambda \Phi''^* \\
 & \left. - 2\sqrt{2} M \Phi''^\dagger \phi^T \Phi'' - 2\sqrt{2} M \Phi'^\dagger \phi \Phi' - 2\sqrt{2} \bar{\Psi}'' P_+ \phi \Psi' \right\},
 \end{aligned} \tag{A.5}$$

$$\begin{aligned}
 \mathcal{L}_2/g^2 = & -2\text{tr} |[\phi, \phi']|^2 - 2\text{tr} |[\phi, \phi'^\dagger]|^2 - 2\text{tr} |[\phi, \phi'']|^2 - 2\text{tr} |[\phi, \phi''^\dagger]|^2 \\
 & - \frac{1}{2} \left| \text{tr} t_a \{ 2[\phi', \phi'^\dagger] - 2[\phi''^\dagger, \phi''] + \Phi' \Phi'^\dagger - \Phi''^* \Phi''^T \} \right|^2 - \text{tr} [\phi^\dagger, \phi]^2 \\
 & - 2 \left| \text{tr} t_a \{ 2[\phi', \phi''] + \Phi' \Phi''^T \} \right|^2 - \Phi'^\dagger \{ \phi, \phi^\dagger \} \Phi' - \Phi''^T \{ \phi, \phi^\dagger \} \Phi''^*.
 \end{aligned} \tag{A.6}$$

Here $P_\pm \equiv \frac{1}{2}(1 \pm \gamma_5)$, t_a are the generators of $\text{SU}(N_c)$ and a sum over a is implied.

The form of the above functions can be greatly simplified upon making the redefinitions (adopted partially from ref. [19])

$$\begin{aligned}
 \psi_1 & \equiv \psi, & \psi_2 & \equiv \lambda, & \psi_3 & \equiv \psi', & \psi_4 & \equiv \psi'', \\
 \omega & \equiv P_+ \Psi' + P_- \Psi'',
 \end{aligned} \tag{A.7}$$

$$\begin{aligned}
 \phi_1 = \phi & = \frac{1}{\sqrt{2}}(X_1 + iY_1), \\
 \phi_2 = \phi' & = \frac{1}{\sqrt{2}}(X_2 + iY_2), \\
 \phi_3 = \phi'' & = \frac{1}{\sqrt{2}}(X_3 + iY_3),
 \end{aligned} \tag{A.8}$$

$$\Phi_1 \equiv \Phi', \quad \Phi_2 \equiv \Phi''^*, \tag{A.9}$$

where X_p and Y_p are hermitian scalar fields and ω a Dirac spinor. It is a straightforward exercise to show that in terms of these variables \mathcal{L}_0 reads

$$\begin{aligned}
 \mathcal{L}_0 = & -\text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_i \not{D} \psi_i + (DX_p)^2 + (DY_p)^2 \right\} \\
 & - \Phi_n^\dagger (-D^2 + M^2) \Phi_n - \bar{\omega} (\not{D} + M) \omega,
 \end{aligned} \tag{A.10}$$

where a summation over repeated indices is implied. Using the Majorana condition of eq. (1.2), the general form of the first six terms in eq. (A.5) can on the other hand be

should be replaced by

$$\begin{aligned}
 & - \frac{1}{4} (\mu^\dagger \mu)_{mn} \phi_m^\dagger \phi_n' - \frac{1}{4} (\mu^\dagger \mu)_{mn} \phi_m''^\dagger \phi_n'' - \text{Re} \mu_{nm} \bar{\psi}_n' P_+ \psi_m'' \\
 & - \sqrt{2} \text{Im} (t'_A \mu)_{mn} \phi_A \phi_n''^* \phi_m'' - \sqrt{2} \text{Im} (\mu t'_A)_{mn} \phi_A \phi_m''^* \phi_n'.
 \end{aligned} \tag{A.2}$$

written as

$$\begin{aligned}
 4\sqrt{2} \operatorname{Im} \operatorname{tr} \left(\bar{\psi}_i P_+ [\phi_k, \psi_j] \right) &= -i2\sqrt{2} \operatorname{tr} \left(\bar{\psi}_i P_+ [\phi_p, \psi_j] - \bar{\psi}_i P_- [\psi_j, \phi_p^\dagger] \right) \\
 &= -2\sqrt{2} \operatorname{tr} \left(i\bar{\psi}_i [\operatorname{Re} \phi_p, \psi_j] - \bar{\psi}_i \gamma_5 [\operatorname{Im} \phi_p, \psi_j] \right), \quad (\text{A.11})
 \end{aligned}$$

which implies that one may simplify their sum into

$$-\operatorname{tr} \left\{ i\bar{\psi}_i \alpha_{ij}^p [X_p, \psi_j] - \bar{\psi}_i \gamma_5 \beta_{ij}^p [Y_p, \psi_j] \right\}. \quad (\text{A.12})$$

Here, α^p and β^p are coefficient matrices that may be taken as antisymmetric as the ψ_i anticommute and whose components can easily be verified to be given by eq. (2.5). The remaining terms in eq. (A.5) are simple to translate into the new variables, leading to the final result

$$\begin{aligned}
 \mathcal{L}_1/g &= \operatorname{tr} \left\{ -i\bar{\psi}_i \alpha_{ij}^p [X_p, \psi_j] + \bar{\psi}_i \gamma_5 \beta_{ij}^p [Y_p, \psi_j] \right\} - \bar{\omega} (Y_1 - i\gamma_5 X_1) \omega \\
 &\quad + 2\sqrt{2} \operatorname{Im} \left\{ -\bar{\omega} P_+ \psi_1 \Phi_1 - \Phi_2^\dagger \bar{\psi}_1 P_+ \omega + \Phi_1^\dagger \bar{\psi}_2 P_+ \omega - \bar{\omega} P_+ \psi_2 \Phi_2 \right\} \\
 &\quad - 2M \Phi_n^\dagger Y_1 \Phi_n. \quad (\text{A.13})
 \end{aligned}$$

Finally attacking \mathcal{L}_2 , the terms in eq. (A.6) that are independent of Φ_n read

$$\begin{aligned}
 &-2\operatorname{tr} |[\phi_1, \phi_2]|^2 - 2\operatorname{tr} |[\phi_1, \phi_2^\dagger]|^2 - 2\operatorname{tr} |[\phi_1, \phi_3]|^2 - 2\operatorname{tr} |[\phi_1, \phi_3^\dagger]|^2 \\
 &- \operatorname{tr} \left| [\phi_2, \phi_2^\dagger] - [\phi_3^\dagger, \phi_3] \right|^2 - \operatorname{tr} [\phi_1^\dagger, \phi_1]^2 - 4\operatorname{tr} |[\phi_2, \phi_3]|^2. \quad (\text{A.14})
 \end{aligned}$$

Using the Jacobi identity for the cross terms on the second line, we have

$$-2\operatorname{tr} [\phi_2, \phi_2^\dagger] [\phi_3, \phi_3^\dagger] = 2\operatorname{tr} |[\phi_2, \phi_3^\dagger]|^2 + 2\operatorname{tr} |[\phi_2, \phi_3]|^2, \quad (\text{A.15})$$

so eq. (A.14) becomes

$$\begin{aligned}
 &-2\operatorname{tr} |[\phi_1, \phi_2]|^2 - 2\operatorname{tr} |[\phi_1, \phi_2^\dagger]|^2 - 2\operatorname{tr} |[\phi_1, \phi_3]|^2 - 2\operatorname{tr} |[\phi_1, \phi_3^\dagger]|^2 \\
 &- 2\operatorname{tr} |[\phi_2, \phi_3]|^2 - 2\operatorname{tr} |[\phi_2, \phi_3^\dagger]|^2 - \operatorname{tr} [\phi_1^\dagger, \phi_1]^2 - \operatorname{tr} [\phi_2^\dagger, \phi_2]^2 - \operatorname{tr} [\phi_3^\dagger, \phi_3]^2 \\
 &= -\frac{1}{2} \operatorname{tr} (i[\chi_A, \chi_B])^2, \quad (\text{A.16})
 \end{aligned}$$

where we have defined $\chi_A \equiv (X_1, Y_1, X_2, Y_2, X_3, Y_3)$. As before, the remaining terms in eq. (A.6) are easy to translate into the new variables, leading to the result

$$\begin{aligned}
 \mathcal{L}_2/g^2 &= -\frac{1}{2} \operatorname{tr} (i[\chi_A, \chi_B])^2 + (-1)^n \Phi_n^\dagger \left([\phi_2, \phi_2^\dagger] + [\phi_3, \phi_3^\dagger] \right) \Phi_n \\
 &\quad - 4\operatorname{Re} \left(\Phi_1^\dagger [\phi_2, \phi_3] \Phi_2 \right) - \frac{1}{2} |(-1)^n \Phi_n^\dagger t_a \Phi_n|^2 - 2|\Phi_2^\dagger t_a \Phi_1|^2 \\
 &\quad - \Phi_n^\dagger \{ \phi_1, \phi_1^\dagger \} \Phi_n, \quad (\text{A.17})
 \end{aligned}$$

where repeated indices are again summed over.

B. Matrix elements and integrals

In this appendix, we will briefly review our evaluation of the necessary scattering amplitudes squared in the non-relativistic limit, as well as explain, how one can perform the integrals in eq. (3.2) analytically in the weak coupling limit. Our treatment is to a large extent parallel to that of ref. [8].

B.1 Matrix elements

As discussed in section 3, the scattering amplitudes squared for the heavy fermions and scalars become identical in the non-relativistic limit, which we exploit by only computing the simpler scalar amplitudes in the Coulomb gauge. We denote the color, flavor and momentum of the initial and final light particles by a, m, \mathbf{k} and b, n, \mathbf{k}' , respectively, and the color and momentum of the heavy particles by i, \mathbf{p} and j, \mathbf{p}' . The angle between \mathbf{k} and \mathbf{k}' is written as θ , the structure constants of the gauge group as f_{abc} and the propagators for the scalars and the gauge field as G_{cd}^p and $D_{cd}^{\mu\nu}$. Because the plasma has no preferred color orientation, we adopt the convention of averaging over the color configurations of the initial heavy particle, while the colors of the light particles as well as the final heavy scalar are summed over.

As a concrete example, consider the amplitude for the process $Sf \rightarrow Sf$. The total scattering amplitude for this process is the sum of the first and fourth diagrams of figure 1.b and is given by

$$\begin{aligned} \mathcal{M}_{Sf \rightarrow Sf} = & \left(-g\delta_{mn}f_{abc}\bar{v}(\mathbf{k})\gamma_\mu v(\mathbf{k}') \right) D_{cd}^{\mu\nu}(Q) \left(ig(P + P')_\nu(t_d)_{ij} \right) \\ & + \left(g\beta_{mn}^1 f_{abc}\bar{v}(\mathbf{k}')\gamma_5 v(\mathbf{k}) \right) G_{cd}^1(Q) \left(-2igM(t_d)_{ij} \right). \end{aligned} \quad (\text{B.1})$$

Upon squaring this expression and summing over m and n , it becomes evident that the cross term will be proportional to $\text{tr} \beta^1$, which vanishes due to the antisymmetry of the matrix. Furthermore, in the non-relativistic limit we have $(P + P')_\nu \approx 2M\delta_{\nu 0}$, so after summing over the colors, flavors and spins of the light fermions as well as the colors of the final heavy scalar, we obtain the result of eq. (3.7a),

$$\begin{aligned} |\mathcal{M}_{Sf \rightarrow Sf}|^2 = & 32g^4 d_A M^2 k^2 (1 + \cos \theta) \frac{1}{(q^2 + m_D^2)^2} \\ & + 32g^4 d_A M^2 k^2 (1 - \cos \theta) \frac{1}{(q^2 + m_S^2)^2}. \end{aligned} \quad (\text{B.2})$$

The results quoted in eqs. (3.7b)–(3.7e) are obtained in a highly analogous fashion.

B.2 Integrals

The integrals appearing in eq. (3.2) are of the same type as those encountered in the QCD case, and our treatment of them therefore follows that of ref. [8] quite closely. We begin by eliminating the three-dimensional delta function through integration over \mathbf{k}' , then change variables from \mathbf{p}' to $\mathbf{q} = \mathbf{p}' - \mathbf{p}$, and finally perform the angular part of the \mathbf{q} integral to

get rid of the energy delta function. This yields

$$\begin{aligned}
 3\kappa_{\text{H}} = & \frac{1}{64\pi^3 M^2} \int_0^\infty dk \int_0^{2k} dq q^3 \left\{ \frac{e^{\beta k}}{(e^{\beta k} + 1)^2} \sum_{f,f'} |\mathcal{M}_{Hf \rightarrow Hf'}|^2 \right. \\
 & \left. + \frac{e^{\beta k}}{(e^{\beta k} - 1)^2} \sum_{b,b'} |\mathcal{M}_{Hb \rightarrow Hb'}|^2 \right\}, \tag{B.3}
 \end{aligned}$$

with $\beta \equiv 1/T$. The sums over the amplitudes squared can be performed using the results of eqs. (3.7a)–(3.7e), and writing the results out explicitly we get

$$\begin{aligned}
 \sum_{f,f'} |\mathcal{M}_{Hf \rightarrow Hf'}|^2 = & 32g^4 d_A M^2 k^2 \left(2 - \frac{q^2}{2k^2} \right) \frac{1}{(q^2 + m_{\text{D}}^2)^2} \\
 & + 32g^4 d_A M^2 k^2 \left(\frac{q^2}{2k^2} \right) \frac{1}{(q^2 + m_{\text{S}}^2)^2}, \tag{B.4}
 \end{aligned}$$

$$\begin{aligned}
 \sum_{b,b'} |\mathcal{M}_{Hb \rightarrow Hb'}|^2 = & 8g^4 d_A M^2 k^2 \left(8 - \frac{q^2}{k^2} + \frac{q^4}{4k^4} \right) \frac{1}{(q^2 + m_{\text{D}}^2)^2} \\
 & + 16g^4 d_A M^2 k^2 \left(\frac{q^2}{k^2} - \frac{q^4}{4k^4} \right) \frac{1}{(q^2 + m_{\text{S}}^2)^2}, \tag{B.5}
 \end{aligned}$$

where the relation $\cos \theta = 1 - q^2/2k^2$ has been applied.

The integral over q in eq. (B.3) can be performed analytically, resulting in a somewhat lengthy one-dimensional integral representation for the diffusion coefficient as a function of the screening masses, which we plotted numerically in section 4. In the true weak coupling limit, where the screening masses satisfy $m/T \sim g \ll 1$, we may, however, simplify the calculation considerably by noting that all terms in eqs. (B.4) and (B.5) proportional to positive powers of q are infrared insensitive and thus independent of the masses to leading order. This enables us to set the masses to zero in these terms and gives

$$\sum_{f,f'} |\mathcal{M}_{Hf \rightarrow Hf'}|^2 = 64g^4 d_A M^2 k^2 \frac{1}{(q^2 + m_{\text{D}}^2)^2}, \tag{B.6}$$

$$\sum_{b,b'} |\mathcal{M}_{Hb \rightarrow Hb'}|^2 = 8g^4 d_A M^2 k^2 \left(\frac{1}{q^2 k^2} - \frac{1}{4k^4} + \frac{8}{(q^2 + m_{\text{D}}^2)^2} \right), \tag{B.7}$$

which simplifies the result of the q integration in eq. (B.3) dramatically. Finally evaluating the k integrals with standard methods, our weak coupling result for $3\kappa_{\text{H}}$ reads

$$\begin{aligned}
 3\kappa_{\text{H}} = & \frac{g^4 d_A}{2\pi^3} \int_0^\infty dk k^2 \left\{ \frac{e^{\beta k}}{(e^{\beta k} + 1)^2} \left(-1 + \log \frac{4k^2}{m_{\text{D}}^2} \right) + \frac{e^{\beta k}}{(e^{\beta k} - 1)^2} \left(-\frac{3}{4} + \log \frac{4k^2}{m_{\text{D}}^2} \right) \right\} \\
 = & \frac{g^4 d_A T^3}{2\pi} \left\{ \log \frac{2T}{m_{\text{D}}} + \frac{13}{12} - \gamma_E + \frac{1}{3} \log 2 + \frac{\zeta'(2)}{\zeta(2)} \right\}, \tag{B.8}
 \end{aligned}$$

which in turn leads to the expression of eq. (3.10) for D_{Q} (or D_{S}).

References

- [1] PHENIX collaboration, S.S. Adler et al., *Nuclear modification of electron spectra and implications for heavy quark energy loss in Au + Au collisions at $\sqrt{S_{NN}} = 200$ GeV*, *Phys. Rev. Lett.* **96** (2006) 032301 [[nucl-ex/0510047](#)].
- [2] STAR collaboration, J. Bielcik, *Centrality dependence of heavy flavor production from single electron measurement in $\sqrt{S_{NN}} = 200$ GeV Au + Au collisions*, *Nucl. Phys. A* **774** (2006) 697 [[nucl-ex/0511005](#)].
- [3] PHENIX collaboration, Y. Akiba, *Probing the properties of dense partonic matter at RHIC*, *Nucl. Phys. A* **774** (2006) 403 [[nucl-ex/0510008](#)].
- [4] M. Djordjevic and M. Gyulassy, *Where is the charm quark energy loss at RHIC?*, *Phys. Lett. B* **560** (2003) 37 [[nucl-th/0302069](#)].
- [5] M. Djordjevic and M. Gyulassy, *Heavy quark radiative energy loss in QCD matter*, *Nucl. Phys. A* **733** (2004) 265 [[nucl-th/0310076](#)].
- [6] Y.L. Dokshitzer and D.E. Kharzeev, *Heavy quark colorimetry of QCD matter*, *Phys. Lett. B* **519** (2001) 199 [[hep-ph/0106202](#)].
- [7] N. Armesto, C.A. Salgado and U.A. Wiedemann, *Medium-induced gluon radiation off massive quarks fills the dead cone*, *Phys. Rev. D* **69** (2004) 114003 [[hep-ph/0312106](#)].
- [8] G.D. Moore and D. Teaney, *How much do heavy quarks thermalize in a heavy ion collision?*, *Phys. Rev. D* **71** (2005) 064904 [[hep-ph/0412346](#)].
- [9] J. Berges and I.O. Stamatescu, *Simulating nonequilibrium quantum fields with stochastic quantization techniques*, *Phys. Rev. Lett.* **95** (2005) 202003 [[hep-lat/0508030](#)].
- [10] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](#)].
- [11] C.P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L.G. Yaffe, *Energy loss of a heavy quark moving through $N = 4$ supersymmetric Yang-Mills plasma*, *JHEP* **07** (2006) 013 [[hep-th/0605158](#)].
- [12] J. Casalderrey-Solana and D. Teaney, *Heavy quark diffusion in strongly coupled $N = 4$ Yang-Mills*, *Phys. Rev. D* **74** (2006) 085012 [[hep-ph/0605199](#)].
- [13] H. Liu, K. Rajagopal and U.A. Wiedemann, *Calculating the jet quenching parameter from AdS/CFT*, [hep-ph/0605178](#).
- [14] C.P. Herzog, *Energy loss of heavy quarks from asymptotically AdS geometries*, *JHEP* **09** (2006) 032 [[hep-th/0605191](#)].
- [15] E. Caceres and A. Guijosa, *Drag force in charged $N = 4$ SYM plasma*, [hep-th/0605235](#).
- [16] J.J. Friess, S.S. Gubser and G. Michalogiorgakis, *Dissipation from a heavy quark moving through $N = 4$ super-Yang-Mills plasma*, *JHEP* **09** (2006) 072 [[hep-th/0605292](#)].
- [17] H. Liu, K. Rajagopal and U.A. Wiedemann, *An AdS/CFT calculation of screening in a Hot Wind*, [hep-ph/0607062](#).
- [18] S. Weinberg, *The quantum theory of fields, vol. 3: Supersymmetry*, Cambridge University Press (2000)

- [19] D. Yamada and L.G. Yaffe, *Phase diagram of $N = 4$ super-Yang-Mills theory with R -symmetry chemical potentials*, *JHEP* **09** (2006) 027 [[hep-th/0602074](#)].
- [20] F. Reif, *Fundamentals of statistical physics*, McGraw-Hill (1965)
- [21] P. Arnold, G.D. Moore and L.G. Yaffe, *Transport coefficients in high temperature gauge theories. I: leading-log results*, *JHEP* **11** (2000) 001 [[hep-ph/0010177](#)].
- [22] A. Denner, H. Eck, O. Hahn and J. Kublbeck, *Compact Feynman rules for Majorana fermions*, *Phys. Lett.* **B 291** (1992) 278.
- [23] E. Braaten and A. Nieto, *Next-to-leading order debye mass for the quark - gluon plasma*, *Phys. Rev. Lett.* **73** (1994) 2402 [[hep-ph/9408273](#)].
- [24] S.C. Huot, S. Jeon and G.D. Moore, *Shear viscosity in weakly coupled $N = 4$ super Yang-Mills theory compared to QCD*, [hep-ph/0608062](#).